

MULTIGRADED FUJITA APPROXIMATION

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ABSTRACT. The original Fujita approximation theorem states that the volume of a big divisor D on a projective variety X can always be approximated arbitrarily closely by the self-intersection number of an ample divisor on a birational modification of X . One can also formulate it in terms of graded linear series as follows: let $W_\bullet = \{W_k\}$ be the complete graded linear series associated to a big divisor D :

$$W_k = H^0(X, \mathcal{O}_X(kD)).$$

For each fixed positive integer p , define $W_\bullet^{(p)}$ to be the graded linear subseries of W_\bullet generated by W_p :

$$W_m^{(p)} = \begin{cases} 0, & \text{if } p \nmid m; \\ \text{Image}(S^k W_p \rightarrow W_{kp}), & \text{if } m = kp. \end{cases}$$

Then the volume of $W_\bullet^{(p)}$ approaches the volume of W_\bullet as $p \rightarrow \infty$. We will show that, under this formulation, the Fujita approximation theorem can be generalized to the case of multigraded linear series.

1. INTRODUCTION

Let X be an irreducible variety of dimension d over an algebraically closed field \mathbf{K} , and let D be a (Cartier) divisor on X . When X is projective, the following limit, which measures how fast the dimension of the section space $H^0(X, \mathcal{O}_X(mD))$ grows, is called the *volume* of D :

$$\text{vol}(D) = \text{vol}_X(D) = \lim_{m \rightarrow \infty} \frac{h^0(X, \mathcal{O}_X(mD))}{m^d/d!}.$$

One says that D is *big* if $\text{vol}(D) > 0$. It turns out that the volume is an interesting numerical invariant of a big divisor ([Laz04, §2.2.C]), and it plays a key role in several recent works in birational geometry ([BDPP04], [Tsu00], [HM06], [Tak06]).

When D is ample, one can show that $\text{vol}(D) = D^d$, the self-intersection number of D . This is no longer true for a general big divisor D , since D^d may even be negative. However, it was shown by Fujita [Fuj94] that the volume of a big divisor can always be approximated arbitrarily closely by the self-intersection number of an ample divisor on a birational modification of X . This theorem, known as *Fujita approximation*, has

Date: 11 May 2010.

2000 Mathematics Subject Classification. 14C20.

Key words and phrases. Fujita approximation, multigraded linear series, Okounkov body.

several implications on the properties of volumes, and is also a crucial ingredient in [BDPP04] (see [Laz04, §11.4] for more details).

In their recent paper [LM08], Lazarsfeld and Mustață obtained, among other things, a generalization of Fujita approximation to *graded linear series*. Recall that a graded linear series $W_\bullet = \{W_k\}$ on a (not necessarily projective) variety X associated to a divisor D consists of finite dimensional vector subspaces

$$W_k \subseteq H^0(X, \mathcal{O}_X(kD))$$

for each $k \geq 0$, with $W_0 = \mathbf{K}$, such that

$$W_k \cdot W_\ell \subseteq W_{k+\ell}$$

for all $k, \ell \geq 0$. Here the product on the left denotes the image of $W_k \otimes W_\ell$ under the multiplication map $H^0(X, \mathcal{O}_X(kD)) \otimes H^0(X, \mathcal{O}_X(\ell D)) \rightarrow H^0(X, \mathcal{O}_X((k+\ell)D))$. In order to state the Fujita approximation for W_\bullet , they defined, for each fixed positive integer p , a graded linear series $W_\bullet^{(p)}$ which is the sub graded linear series of W_\bullet generated by W_p :

$$W_m^{(p)} = \begin{cases} 0, & \text{if } p \nmid m; \\ \text{Im}(S^k W_p \rightarrow W_{kp}), & \text{if } m = kp. \end{cases}$$

Then under mild hypotheses, they showed that the volume of $W_\bullet^{(p)}$ approaches the volume of W_\bullet as $p \rightarrow \infty$. See [LM08, Theorem 3.5] for the precise statement, as well as [LM08, Remark 3.4] for how this is equivalent to the original statement of Fujita when X is projective and W_\bullet is the complete graded linear series associated to a big divisor D (i.e. $W_k = H^0(X, \mathcal{O}_X(kD))$ for all $k \geq 0$).

The goal of this note is to generalize the Fujita approximation theorem to *multigraded linear series*. We will adopt the following notations from [LM08, §4.3]: let D_1, \dots, D_r be divisors on X . For $\vec{m} = (m_1, \dots, m_r) \in \mathbb{N}^r$ we write $\vec{m}D = \sum m_i D_i$, and we put $|\vec{m}| = \sum |m_i|$.

Definition 1. A *multigraded linear series* W_\bullet on X associated to the D_i 's consists of finite-dimensional vector subspaces

$$W_{\vec{k}} \subseteq H^0(X, \mathcal{O}_X(\vec{k}D))$$

for each $\vec{k} \in \mathbb{N}^r$, with $W_{\vec{0}} = \mathbf{K}$, such that

$$W_{\vec{k}} \cdot W_{\vec{m}} \subseteq W_{\vec{k}+\vec{m}},$$

where the multiplication on the left denotes the image of $W_{\vec{k}} \otimes W_{\vec{m}}$ under the natural map $H^0(X, \mathcal{O}_X(\vec{k}D)) \otimes H^0(X, \mathcal{O}_X(\vec{m}D)) \rightarrow H^0(X, \mathcal{O}_X((\vec{k}+\vec{m})D))$.

Given $\vec{a} \in \mathbb{N}^r$, denote by $W_{\vec{a}, \bullet}$ the singly graded linear series associated to the divisor $\vec{a}D$ given by the subspaces $W_{k\vec{a}} \subseteq H^0(X, \mathcal{O}_X(k\vec{a}D))$. Then put

$$\text{vol}_{W_\bullet}(\vec{a}) = \text{vol}(W_{\vec{a}, \bullet})$$

(assuming that this quantity is finite). It will also be convenient for us to consider $W_{\vec{a}, \bullet}$ when $\vec{a} \in \mathbb{Q}_{\geq 0}^r$, given by

$$W_{\vec{a}, k} = \begin{cases} W_{k\vec{a}}, & \text{if } k\vec{a} \in \mathbb{N}^r; \\ 0, & \text{otherwise.} \end{cases}$$

Our multigraded Fujita approximation, similar to the singly-graded version, is going to state that (under suitable conditions) the volume of W_{\bullet} can be approximated by the volume of the following finitely generated sub multigraded linear series of W_{\bullet} :

Definition 2. Given a multigraded linear series W_{\bullet} and a positive integer p , define $W_{\bullet}^{(p)}$ to be the sub multigraded linear series of W_{\bullet} generated by all $W_{\vec{m}_i}$ with $|\vec{m}_i| = p$, or concretely

$$W_{\vec{m}}^{(p)} = \begin{cases} 0, & \text{if } p \nmid |\vec{m}|; \\ \sum_{\substack{|\vec{m}_i|=p \\ \vec{m}_1 + \dots + \vec{m}_k = \vec{m}}} W_{\vec{m}_1} \cdots W_{\vec{m}_k}, & \text{if } |\vec{m}| = kp. \end{cases}$$

We now state our multigraded Fujita approximation when W_{\bullet} is a complete multigraded linear series, since this is the case of most interest and allows for a more streamlined statement. We will point out in Remark 4 afterward what assumptions on W_{\bullet} are actually needed in the proof.

Theorem 3. *Let X be an irreducible projective variety of dimension d , and let D_1, \dots, D_r be big divisors on X . Let W_{\bullet} be the complete multigraded linear series associated to the D_i 's, namely*

$$W_{\vec{m}} = H^0(X, \mathcal{O}_X(\vec{m}D))$$

for each $\vec{m} \in \mathbb{N}^r$. Then given any $\varepsilon > 0$, there exists an integer $p_0 = p_0(\varepsilon)$ having the property that if $p \geq p_0$, then

$$(1) \quad \left| 1 - \frac{\text{vol}_{W_{\bullet}^{(p)}}(\vec{a})}{\text{vol}_{W_{\bullet}}(\vec{a})} \right| < \varepsilon$$

for all $\vec{a} \in \mathbb{N}^r$.

Acknowledgments. The author would like to thank Robert Lazarsfeld for raising this question during an email correspondence.

2. PROOF OF THEOREM 3

The main tool in our proof is the theory of *Okounkov bodies* developed systematically in [LM08]. Given a graded linear series W_{\bullet} on a d -dimensional variety X , its Okounkov body $\Delta(W_{\bullet})$ is a convex body in \mathbb{R}^d that encodes many asymptotic invariants of W_{\bullet} , the most prominent one being the volume of W_{\bullet} , which is precisely

$d!$ times the Euclidean volume of $\Delta(W_\bullet)$. The idea first appeared in Okounkov's papers [Oko96] and [Oko03] in the case of complete linear series of ample line bundles on a projective variety. Later it was further developed and applied to much more general graded linear series by Lazarsfeld-Mustață [LM08], and also independently by Kaveh-Khovanskii [KK08, KK09].

Proof of Theorem 3. Let $T = \{(a_1, \dots, a_r) \in \mathbb{R}_{\geq 0}^r \mid a_1 + \dots + a_r = 1\}$, and let $T_{\mathbb{Q}}$ be the set of all points in T with rational coordinates. The fraction inside (1) is invariant under scaling of \vec{a} due to homogeneity, hence it is enough to prove (1) for $\vec{a} \in T_{\mathbb{Q}}$.

Let $\Delta(W_\bullet) \subseteq \mathbb{R}^d \times \mathbb{R}^r$ be the global Okounkov cone of W_\bullet as in [LM08, Theorem 4.19], and let $\pi: \Delta(W_\bullet) \rightarrow \mathbb{R}^r$ be the projection map. For each $\vec{a} \in T$ we write $\Delta(W_\bullet)_{\vec{a}}$ for the fiber $\pi^{-1}(\vec{a})$. We also define in a similar fashion the convex cone $\Delta(W_\bullet^{(p)})$ and the convex bodies $\Delta(W_\bullet^{(p)})_{\vec{a}}$. By [LM08, Theorem 4.19],

$$(2) \quad \Delta(W_\bullet)_{\vec{a}} = \Delta(W_{\vec{a}, \bullet}) \quad \text{for all } \vec{a} \in T_{\mathbb{Q}}.$$

Note that although [LM08, Theorem 4.19] requires \vec{a} to be in the relative interior of T , here we know that (2) holds even for those \vec{a} in the boundary of T because the big cone of X is open and W_\bullet was assumed to be the complete multigraded linear series. By the singly-graded Fujita approximation, $\text{vol}(W_{\vec{a}, \bullet})$ can be approximated arbitrarily closely by $\text{vol}(W_{\vec{a}, \bullet}^{(p)})$ if p is sufficiently large. (Here by $W_{\vec{a}, \bullet}^{(p)}$ we mean $W_\bullet^{(p)}$ restricted to the \vec{a} direction, which certainly contains $(W_{\vec{a}, \bullet})^{(p)}$.) Hence given any finite subset $S \subset T_{\mathbb{Q}}$ and any $\varepsilon' > 0$, we have

$$\text{vol}(\Delta(W_\bullet^{(p)})_{\vec{a}}) \geq \text{vol}(\Delta(W_\bullet)_{\vec{a}}) - \varepsilon' \quad \text{for all } \vec{a} \in S$$

as soon as p is sufficiently large.

Because the function $\vec{a} \mapsto \text{vol}(\Delta(W_\bullet)_{\vec{a}})$ is uniformly continuous on T , given any $\varepsilon' > 0$, we can partition T into a union of polytopes with disjoint interiors $T = \bigcup T_i$, in such a way that the vertices of each T_i all have rational coordinates, and on each T_i we have a constant M_i such that

$$(3) \quad M_i \leq \text{vol}(\Delta(W_\bullet)_{\vec{a}}) \leq M_i + \varepsilon' \quad \text{for all } \vec{a} \in T_i.$$

Let S be the set of vertices of all the T_i 's. Then as we saw in the end of the previous paragraph, as soon as p is sufficiently large we have

$$(4) \quad \text{vol}(\Delta(W_\bullet^{(p)})_{\vec{a}}) \geq \text{vol}(\Delta(W_\bullet)_{\vec{a}}) - \varepsilon' \quad \text{for all } \vec{a} \in S.$$

We claim that this implies

$$(5) \quad \text{vol}(\Delta(W_\bullet^{(p)})_{\vec{a}}) \geq \text{vol}(\Delta(W_\bullet)_{\vec{a}}) - 2\varepsilon' \quad \text{for all } \vec{a} \in T_{\mathbb{Q}}.$$

To show this, it suffices to verify it on each of the T_i 's. Let $\vec{v}_1, \dots, \vec{v}_k$ be the vertices of T_i . Then each $\vec{a} \in T_i$ can be written as a convex combination of the vertices:

$\vec{a} = \sum t_j \vec{v}_j$ where each $t_j \geq 0$ and $\sum t_j = 1$. Since $\Delta(W_{\bullet}^{(p)})$ is convex, we have

$$\Delta(W_{\bullet}^{(p)})_{\vec{a}} \supseteq \sum t_j \Delta(W_{\bullet}^{(p)})_{\vec{v}_j},$$

where the sum on the right means the Minkowski sum. By (3) and (4), the volume of each $\Delta(W_{\bullet}^{(p)})_{\vec{v}_j}$ is at least $M_i - \varepsilon'$, hence by the Brunn-Minkowski inequality [KK08, Theorem 5.4], we have

$$\text{vol}(\Delta(W_{\bullet}^{(p)})_{\vec{a}}) \geq M_i - \varepsilon' \quad \text{for all } \vec{a} \in T_i \cap T_{\mathbb{Q}}.$$

This combined with (3) shows that (5) is true on $T_i \cap T_{\mathbb{Q}}$, hence it is true on $T_{\mathbb{Q}}$ since the T_i 's cover T .

Since (1) follows from (5) by choosing a suitable ε' , the proof is thus complete. \square

Remark 4. In the statement of Theorem 3 we assume that W_{\bullet} is the complete multigraded linear series associated to big divisors. But in fact since the main tool we used in the proof is the theory of Okounkov bodies established in [LM08], in particular [LM08, Theorem 4.19], the really indispensable assumptions on W_{\bullet} are the same as those in [LM08] (which they called Conditions (A') and (B'), or (C')). The only place in the proof where we invoke that we are working with a complete multigraded linear series is the sentence right after (2), where we want to say that (2) holds not only in the relative interior of T but also in its boundary. Hence if W_{\bullet} is only assumed to satisfy Conditions (A') and (B'), or (C'), then given any $\varepsilon > 0$ and any compact set C contained in $T \cap \text{int}(\text{supp}(W_{\bullet}))$, there exists an integer $p_0 = p_0(C, \varepsilon)$ such that if $p \geq p_0$ then

$$\text{vol}_{W_{\bullet}^{(p)}}(\vec{a}) > \text{vol}_{W_{\bullet}}(\vec{a}) - \varepsilon$$

for all $\vec{a} \in C \cap T_{\mathbb{Q}}$.

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